You should work 10 problems a day, 7 days a week, until you finish all of these problems

Techniques of Integration

MISCELLANEOUS PROBLEMS

The students really should work most of these problems over a period of several days, even while you continue to later chapters. Particularly interesting problems in this set include 23, 37, 39, 60, 78, 79, 83, 94, 100, 102, 110 and 111 together, 115, 117, and 119.

Evaluate the integrals in Problems 1–100.

1. $\int \frac{1}{\sqrt{x}(1+x)}\,dx$
2. $\int \frac{\sec^2 t}{1+\tan t}\,dt$
3. $\int \sin x \sec x\,dx$
4. $\int \frac{\csc x \cot x}{1+\csc^2 x}\,dx$
5. $\int \frac{\tan\theta}{\cos^2\theta}\,d\theta$
6. $\int \csc^4 x\,dx$
7. $\int x\tan^2 x\,dx$
8. $\int x^2\cos^2 x\,dx$
9. $\int x^5\sqrt{2-x^3}\,dx$
10. $\int \frac{1}{\sqrt{x^2+4}}\,dx$
11. $\int \frac{x^2}{\sqrt{25+x^2}}\,dx$
12. $\int (\cos x)\sqrt{4-\sin^2 x}\,dx$
13. $\int \frac{1}{x^2-x+1}\,dx$
14. $\int \sqrt{x^2+x+1}\,dx$
15. $\int \frac{5x+31}{3x^2-4x+11}\,dx$
16. $\int \frac{x^4+1}{x^2+2}\,dx$
17. $\int \frac{1}{5+4\cos\theta}\,d\theta$
18. $\int \frac{\sqrt{x}}{1+x}\,dx$
19. $\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx$
20. $\int \frac{\cos 2x}{\cos x}\,dx$
21. $\int \frac{\tan x}{\ln(\cos x)}\,dx$
22. $\int \frac{x^7}{\sqrt{1-x^4}}\,dx$
23. $\int \ln(1+x)\,dx$
24. $\int x\sec^{-1} x\,dx$
25. $\int \sqrt{x^2+9}\,dx$
26. $\int \frac{x^2}{\sqrt{4-x^2}}\,dx$
27. $\int \sqrt{2x-x^2}\,dx$
28. $\int \frac{4x-2}{x^3-x}\,dx$
29. $\int \frac{x^4}{x^2-2}\,dx$
30. $\int \frac{\sec x\tan x}{\sec x+\sec^2 x}\,dx$
31. $\int \frac{x}{(x^2+2x+2)^2}\,dx$
32. $\int \frac{x^{1/3}}{x^{1/2}+x^{1/4}}\,dx$
33. $\int \frac{1}{1+\cos 2\theta}\,d\theta$
34. $\int \frac{\sec x}{\tan x}\,dx$
35. $\int \sec^3 x\tan^3 x\,dx$
36. $\int x^2\tan^{-1} x\,dx$
37. $\int x(\ln x)^3\,dx$
38. $\int \frac{1}{x\sqrt{1+x^2}}\,dx$
39. $\int e^x\sqrt{1+e^{2x}}\,dx$
40. $\int \frac{x}{\sqrt{4x-x^2}}\,dx$
41. $\int \frac{1}{x^3\sqrt{x^2-9}}\,dx$
42. $\int \frac{x}{(7x+1)^{17}}\,dx$
43. $\int \frac{4x^2+x+1}{4x^3+x}$
44. $\int \frac{4x^3-x+1}{x^3+1}\,dx$
45. $\int \tan^2 x\sec x\,dx$
46. $\int \frac{x^2+2x+2}{(x+1)^3}\,dx$
47. $\int \frac{x^4+2x+2}{x^5+x^4}\,dx$
48. $\int \frac{8x^2-4x+7}{(x^2+1)(4x+1)}\,dx$
49. $\int \frac{3x^5-x^4+2x^3-12x^2-2x+1}{(x^3-1)^2}\,dx$
50. $\int \frac{x}{x^4+4x^2+8}\,dx$
51. $\int \frac{1}{4+5\cos\theta}\,d\theta$

52. $\int \frac{(1+x^{2/3})^{3/2}}{x^{1/3}}\,dx$

53. $\int \frac{(\arcsin x)^2}{\sqrt{1-x^2}}\,dx$

54. $\int \frac{1}{x^{3/2}(1+x^{1/3})}\,dx$

55. $\int \tan^3 z\,dz$

56. $\int \sin^2\omega\cos^4\omega\,d\omega$

57. $\int \frac{xe^{x^2}}{1+e^{2x^2}}\,dx$

58. $\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$

59. $\int x^3\exp(-x^2)\,dx$

60. $\int \sin\sqrt{x}\,dx$

61. $\int \frac{\arcsin x}{x^2}\,dx$

62. $\int \sqrt{x^2-9}\,dx$

63. $\int x^2\sqrt{1-x^2}\,dx$

64. $\int x\sqrt{2x-x^2}\,dx$

65. $\int \frac{x-2}{4x^2+4x+1}\,dx$

66. $\int \frac{2x^2-5x-1}{x^3-2x^2-x+2}\,dx$

67. $\int \frac{e^{2x}}{e^{2x}-1}\,dx$

68. $\int \frac{\cos x}{\sin^2 x-3\sin x+2}\,dx$

69. $\int \frac{2x^3+3x^2+4}{(x+1)^4}\,dx$

70. $\int \frac{\sec^2 x}{\tan^2 x+2\tan x+2}\,dx$

71. $\int \frac{x^3+x^2+2x+1}{x^4+2x^2+1}\,dx$

72. $\int \frac{3+\cos\theta}{2-\cos\theta}\,d\theta$

73. $\int x^5\sqrt{x^3-1}\,dx$

74. $\int \frac{1}{2+2\cos\theta+\sin\theta}\,d\theta$

75. $\int \frac{\sqrt{1+\sin x}}{\sec x}\,dx$

76. $\int \frac{1}{x^{2/3}(1+x^{2/3})}\,dx$

77. $\int \frac{\sin x}{\sin 2x}\,dx$

78. $\int \sqrt{1+\cos t}\,dt$

79. $\int \sqrt{1+\sin t}\,dt$

80. $\int \frac{\sec^2 t}{1-\tan^2 t}\,dt$

81. $\int \ln(x^2+x+1)\,dx$

82. $\int e^x\sin^{-1}(e^x)\,dx$

83. $\int \frac{\arctan x}{x^2}\,dx$

84. $\int \frac{x^2}{\sqrt{x^2-25}}\,dx$

85. $\int \frac{x^3}{(x^2+1)^2}\,dx$

86. $\int \frac{1}{x\sqrt{6x-x^2}}\,dx$

87. $\int \frac{3x+2}{(x^2+4)^{3/2}}\,dx$

88. $\int x^{3/2}\ln x\,dx$

89. $\int \frac{(1+\sin^2 x)^{1/2}}{\sec x\csc x}\,dx$

90. $\int \frac{\exp(\sqrt{\sin x})}{(\sec x)\sqrt{\sin x}}\,dx$

91. $\int xe^x\sin x\,dx$

92. $\int x^2\exp(x^{3/2})\,dx$

93. $\int \frac{\arctan x}{(x-1)^3}\,dx$

94. $\int \ln(1+\sqrt{x})\,dx$

95. $\int \frac{2x+3}{\sqrt{3+6x-9x^2}}\,dx$

96. $\int \frac{1}{2+2\sin\theta+\cos\theta}\,d\theta$

97. $\int \frac{\sin^3\theta}{\cos\theta-1}\,d\theta$

98. $\int x^{3/2}\tan^{-1}(x^{1/2})\,dx$

99. $\int \sec^{-1}\sqrt{x}\,dx$

100. $\int x\left(\frac{1-x^2}{1+x^2}\right)^{1/2}dx$

101. Find the area of the surface generated by revolving the curve $y=\cosh x$, $0\leqq x\leqq 1$, around the x-axis.

102. Find the length of the curve $y=e^{-x}$, $0\leqq x\leqq 1$.

103. (a) Find the area A_b of the surface generated by revolving the curve $y=e^{-x}$, $0\leqq x\leqq b$, around the x-axis.
(b) Find $\lim_{b\to\infty} A_b$.

104. (a) Find the area A_b of the surface generated by revolving the curve $y=1/x$, $1\leqq x\leqq b$, around the x-axis.
(b) Find $\lim_{b\to\infty} A_b$.

105. Find the area of the surface generated by revolving the curve

$$y=\sqrt{x^2-1},\qquad 1\leqq x\leqq 2,$$

around the x-axis.

106. (a) Derive the reduction formula

$$\int x^m(\ln x)^n\,dx=\frac{1}{m+1}x^{m+1}(\ln x)^n-\frac{n}{m+1}\int x^m(\ln x)^{n-1}\,dx.$$

(b) Evaluate $\int_1^e x^3(\ln x)^3\,dx$.

107. Derive the reduction formula

$$\int \sin^m x\cos^n x\,dx=-\frac{1}{m+n}\sin^{m-1}x\cos^{n+1}x+\frac{m-1}{m+n}\int \sin^{m-2}x\cos^n x\,dx.$$

108. Use the reduction formula of Problem 107 here and Problem 46 in Section 9.4 to evaluate

$$\int_0^{\pi/2} \sin^6 x \cos^5 x \, dx.$$

109. Find the area bounded by the curve $y^2 = x^5(2 - x)$, $0 \leqq x \leqq 2$. (*Suggestion:* Substitute $x = 2\sin^2\theta$, and then use the result of Problem 50 in Section 9.4.)

110. Show that

$$\int_0^1 \frac{t^4(1-t)^4}{1+t^2}\, dt = \frac{22}{7} - \pi.$$

111. Evaluate

$$\int_0^1 t^4(1-t)^4 \, dt,$$

then apply the result of Problem 110 to conclude that

$$\tfrac{22}{7} - \tfrac{1}{630} < \pi < \tfrac{22}{7} - \tfrac{1}{1260}.$$

Thus $3.1412 < \pi < 3.1421$.

112. Find the length of the curve $y = \frac{4}{5}x^{5/4}$, $0 \leqq x \leqq 1$.

113. Find the length of the curve $y = \frac{4}{3}x^{3/4}$, $1 \leqq x \leqq 4$.

114. An initially empty water tank is shaped like a cone with vertical axis, vertex at the bottom, 9 ft deep, and top radius 4.5 ft. Beginning at time $t = 0$, water is poured into this tank at 50 ft^3/min. Meanwhile, water leaks out a hole at the bottom at the rate of $10\sqrt{y}$ cubic feet per minute where y is the depth of the water in the tank. (This is consistent with Torricelli's law.) How long does it take to fill the tank?

115. (a) Evaluate $\displaystyle\int \frac{1}{1 + e^x + e^{-x}}\, dx.$

(b) Explain why your substitution in (a) suffices to integrate any rational function of e^x.

116. (a) The equation $x^3 + x + 1 = 0$ has one real root r. Use Newton's method to find it, accurate to at least two places.

(b) Use long division to find the irreducible quadratic factor of $x^3 + x + 1$.

(c) Use the factorization of part (b) to evaluate

$$\int_0^1 \frac{1}{x^3 + x + 1}\, dx.$$

117. Evaluate $\displaystyle\int \frac{1}{1 + e^x}\, dx.$

118. The integral

$$\int \frac{1 + 2x^2}{x^5(1 + x^2)^3}\, dx = \int \frac{x + 2x^3}{(x^4 + x^2)^3}\, dx$$

would require solving eleven equations in eleven unknowns if the method of partial fractions were used to evaluate it. Use the substitution $u = x^4 + x^2$ to evaluate it much more simply.

119. Evaluate

$$\int \sqrt{\tan\theta}\, d\theta.$$

Suggestion: First substitute $u = \tan\theta$. Then substitute $u = x^2$. Finally use the method of partial fractions; see Problem 42 in Section 9.7.)

Be able to do this Example for Exam 2.

EXAMPLE 6 Find a reduction formula for $\int \sec^n x\, dx$.

Solution The idea is that n is a (large) positive integer, and that we want to express the given integral in terms of a lower power of sec x. The easiest power of sec x to integrate is $\sec^2 x$, so we proceed as follows.

We don't choose $dv = \sec x\, dx$ because this would introduce a natural logarithm function, a fearsome complication in the second integration.

Let $\quad u = \sec^{n-2} x, \qquad dv = \sec^2 x\, dx.$

Then $\quad du = (n-2)\sec^{n-2} x \tan x\, dx, \qquad v = \tan x.$

This gives

$$\int \sec^n x\, dx = \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int (\sec^{n-2} x)(\sec^2 x - 1)\, dx.$$

Hence

$$\int \sec^n x\, dx = \sec^{n-2} x \tan x - (n-2)\int \sec^n x\, dx + (n-2)\int \sec^{n-2} x\, dx.$$

We solve this equation for the desired integral and find that

$$\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\, dx. \tag{5}$$

This is the desired reduction formula. For example, if we take $n = 3$ in this formula, we find that

$$\int \sec^3 x\, dx = \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\int \sec x\, dx$$

$$= \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln|\sec x + \tan x| + C. \tag{6}$$

In the last step we used Equation (15) of Section 8.2,

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C.$$

The reason for using the reduction formula in (5) is that repeated application must yield one of the two elementary integrals $\int \sec x\, dx$ and $\int \sec^2 x\, dx$. For instance, with $n = 4$ we get

$$\int \sec^4 x\, dx = \tfrac{1}{3}\sec^2 x \tan x + \tfrac{2}{3}\int \sec^2 x\, dx$$

$$= \tfrac{1}{3}\sec^2 x \tan x + \tfrac{2}{3}\tan x + C, \tag{7}$$

and with $n = 5$ we get

$$\int \sec^5 x\, dx = \tfrac{1}{4}\sec^3 x \tan x + \tfrac{3}{4}\int \sec^3 x\, dx$$

$$= \tfrac{1}{4}\sec^3 x \tan x + \tfrac{3}{8}\sec x \tan x + \tfrac{3}{8}\ln|\sec x + \tan x| + C, \tag{8}$$

using in the last step the formula in Equation (6).