Summary of Integration Techniques

When I look at evaluating an integral, I think through the following strategies.

1. **Basic Integral?** Do a little algebra and simplify.

2. **Substitutions**

   (a) u-substitution: u equals to a part of the function you are integrating

   (b) Trigonometric Substitution:

   $a^2 + b^2x^2$ use $x = \frac{a}{b}\tan\theta$
   $a^2 - b^2x^2$ use $x = \frac{a}{b}\sin\theta$
   $b^2x^2 - a^2$ use $x = \frac{a}{b}\sec\theta$

3. **Trigonometric Integrals**

   (a) Integrating $\sin^n x \cos^m x$

   - Odd power of sine or cosine: use u-substitution.
   - Both powers are even: use half-angle identities

   (b) Other trigonometric integrals

   - Try to factor out a $\sec x \tan x$ or $\sec^2 x$ and do a u-substitution
     - If $u = \tan x$, then $du = \sec^2 x\,dx$.
     - If $u = \sec x$, then $du = \sec x \tan x\,dx$.
   - Try to write everything in terms of sine and cosine

   (c) Use some identities:

   $\sin^2 x + \cos^2 x = 1$

   $\tan^2 x + 1 = \sec^2 x$

   $\sin 2x = 2 \sin x \cos x$

   $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

   $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

   $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

   We did not prove this, but this may come up in exercises so you are given that:

   $\int \sec x\,dx = \ln|\sec x + \tan x| + C$ and $\int \csc x\,dx = \ln|\csc x - \cot x| + C$

4. **Partial Fractions:** Given $f(x) = \dfrac{P(x)}{Q(x)}$ where P and Q are polynomials, we want to write f as the sum of simpler fractions. (This type is easy to identify.)

   (a) If improper, then divide. (If $\deg(P(x)) \geq \deg(Q(x))$)

   (b) $Q(x)$ has a factor like ..., then the partial fraction decomposition must include ... .

$$(px+q) \quad \to \quad \frac{A}{px+q}$$

$$(px+q)^m \quad \to \quad \frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \cdots + \frac{A_n}{(px+q)^m}$$

$$(ax^2+bx+c) \quad \to \quad \frac{Bx+c}{ax^2+bx+c}$$

$$(ax^2+bx+c)^n \quad \to \quad \frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{(ax^2+bx+c)^2} + \cdots + \frac{B_nx+C_n}{(ax^2+bx+c)^n}$$

   Solve for the constants by multiplying both sides of the equation by the denominator, then set x to convenient values.

5. **Integration by Parts:** $I = \int u\,dv = uv - \int v\,du$

   (a) $x^n f(x)$ where f is $\sin(ax), \cos(ax)$, or e^{ax} for some constant a $\to$ use tabular method

   (b) exponential $\times$ [$\sin(ax)$ or $\cos(ax)$] $\to$ integrate by parts twice to get the same integral type on both sides of the equation and solve for the integral

   (c) Just $\sin^{-1} x$ or $\ln x \Rightarrow dv = dx$

6. *Might* be useful: Given $\sqrt[n]{g(x)}$, try $u = \sqrt[n]{g(x)}$ so that $u^n = g(x)$, $nu^{n-1} = g'(x)\,dx$

Note that I typically think of integration by parts last. Also, one gets better through practice.

So work out lots of problems. Happy Studying!!