Mathematics Learning Centre

The University of Sydney

Introduction to Integration
Part 1: Anti-Differentiation

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Contents

1 For Reference

1.1 Table of derivatives

Function $(f(x))$	**Derivative** $\left(f'(x) \text{ i.e. } \frac{d}{dx}(f(x))\right)$	
x^n	nx^{n-1}	(n any real number)
e^x	e^x	
$\ln x$	$\frac{1}{x}$	$(x > 0)$
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	
$\cot x$	$-\csc^2 x$	
$\sec x$	$\sec x \tan x$	
$\csc x$	$-\csc x \cot x$	
$\sin^{-1} x$	$\dfrac{1}{\sqrt{(1-x^2)}}$	$(\lvert x\rvert < 1)$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	

1.2 New notation

Symbol	**Meaning**
$\int f(x)dx$	The indefinite integral of $f(x)$ with respect to x i.e. a function whose derivative is $f(x)$.

Note that $\int ...dx$ acts like a pair of brackets around the function. Just as a left-hand bracket has no meaning unless it is followed by a closing right-hand bracket, the integral sign cannot stand by itself, but needs "dx" to complete it. The integral sign tells us what operation to perform and the "dx" tells us that the variable with respect to which we are integrating is x.

New terms	**Meaning**
Anti-derivative Primitive function Indefinite integral	These are all different ways of saying "a function whose derivative is ..."

2 Introduction

This booklet is intended for students who have never done integration before, or who *have* done it before, but so long ago that they feel they have forgotten it all.

Integration is used in dealing with two essentially different types of problems:

The first type are problems in which the derivative of a function, or its rate of change, or the slope of its graph, is known and we want to find the function. We are therefore required to *reverse* the process of differentiation. This reverse process is known as *anti-differentiation*, or *finding a primitive function*, or *finding an indefinite integral*.

The second type are problems which involve adding up a very large number of very small quantities, (and then taking a limit as the size of the quantities approaches zero while the number of terms tends to infinity). This process leads to the definition of the *definite integral*. Definite integrals are used for finding area, volume, centre of gravity, moment of inertia, work done by a force, and in many other applications.

This unit will deal only with problems of the first type, i.e. with indefinite integrals. The second type of problem is dealt with in *Introduction to Integration Part 2 - The Definite Integral*.

2.1 How to use this book

You will not gain much by just reading this booklet. Have pencil and paper ready to work through the examples before reading their solutions. Do *all* the exercises. It is important that you try hard to complete the exercises on your own, rather than refer to the solutions as soon as you are stuck. If you have done integration before, and want to revise it, you should skim through the text and then do the exercises for practice. If you have any difficulties with the exercises, go back and study the text in more detail.

2.2 Objectives

By the time you have worked through this unit, you should:

- Be familiar with the definition of an indefinite integral as the result of reversing the process of differentiation.
- Understand how rules for integration are worked out using the rules for differentiation (in reverse).
- Be able to find indefinite integrals of sums, differences and constant multiples of certain elementary functions.
- Be able to use the chain rule (in reverse) to find indefinite integrals of certain expressions involving composite functions.
- Be able to apply these techniques to problems in which the rate of change of a function is known and the function has to be found.

2.3 Assumed knowledge

We assume that you are familiar with the following elementary functions: polynomials, powers of x, and the trigonometric, exponential and natural logarithm functions, and are able to differentiate these. We also assume that you can recognise composite functions and are familiar with the chain rule for differentiating them.

In addition you will need to know some simple trigonometric identities: those based on the definitions of tan, cot, sec and csc and those based on Pythagoras' Theorem. These are covered in sections of the Mathematics Learning Centre booklet *Trigonometric Identities*.

Other trigonometric identities are not needed for this booklet, but will be needed in any course on integration, so if you are preparing for a course on integration you should work through the whole of *Trigonometric Identities* as well as this booklet.

Finally, knowledge of the inverse trigonometric functions, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ and their derivatives would be a help, but is not essential.

2.4 Test yourself

To check how well you remember all the things we will be assuming, try the following questions, and check your answers against those on the next page.

1. Find the derivatives of the following functions:

i x^{10}

ii $\sqrt{x}$

iii $\dfrac{1}{x}$

iv $5x^3 - \frac{3}{x^2}$

2. Find $f'(x)$ for each of the functions $f(x)$ given:

i $f(x) = e^x$

ii $f(x) = \ln(x)$

iii $f(x) = \cos x + \sin x$

iv $f(x) = \cot x$

3. Find derivatives of:

i $(2x + 1)^{12}$

ii $\sin 3x$

iii e^{x^2}

iv $\dfrac{1}{x^2 - 3}$

v $\cos(x^3)$

vi $\ln(\sin x)$

4. Simplify the following expressions:

i $\tan x \csc x$

ii $1 - \sec^2 x$

***5.** Find the derivatives of:

i $\sin^{-1} x$

ii $\tan^{-1} x$

*Omit this question if you have not studied inverse trigonometric functions.

2.5 Solutions to 'Test yourself'

1. **i** $\frac{d}{dx}(x^{10}) = 10x^9$

ii $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

iii $\frac{d}{dx}(\frac{1}{x}) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$

iv $\frac{d}{dx}(5x^3 - \frac{3}{x^2}) = 5 \cdot 3x^2 - 3 \cdot (-2)x^{-3} = 15x^2 + \frac{6}{x^3}$

2. **i** $f'(x) = e^x$

ii $f'(x) = \frac{1}{x}$

iii $f'(x) = -\sin x + \cos x$

iv $f'(x) = -\csc^2 x$

3. **i** $\frac{d}{dx}(2x+1)^{12} = 12(2x+1)^{11} \cdot 2 = 24(2x+1)^{11}$

ii $\frac{d}{dx}(\sin 3x) = \cos 3x \cdot 3 = 3\cos 3x$

iii $\frac{d}{dx}(e^{x^2}) = e^{x^2} \cdot 2x = 2xe^{x^2}$

iv $\frac{d}{dx}(\frac{1}{x^2-3}) = \frac{d}{dx}((x^2-3)^{-1}) = (-1)(x^2-3)^{-2} \cdot 2x = -\frac{2x}{(x^2-3)^2}$

v $\frac{d}{dx}(\cos(x^3)) = -\sin(x^3) \cdot 3x^2 = -3x^2\sin(x^3)$

vi $\frac{d}{dx}(\ln(\sin x)) = \frac{1}{\sin x} \cdot \cos x = \cot x$

4. **i** $\tan x \csc x = \frac{\sin x}{\cos x} \cdot \frac{1}{\sin x} = \frac{1}{\cos x} = \sec x$

ii Since $\sec^2 x = 1 + \tan^2 x$, $1 - \sec^2 x = -\tan^2 x$

***5.** **i** $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

ii $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

If you had difficulty with many of these questions it may be better for you to revise differentiation and trig identities before going on with this booklet.

3 Definition of the Integral as an Anti-Derivative

If $\dfrac{d}{dx}(F(x)) = f(x)$ then $\int f(x)dx = F(x)$.

In words,

> If the derivative of $F(x)$ is $f(x)$, then we say that an indefinite integral of $f(x)$ with respect to x is $F(x)$.

For example, since the derivative (with respect to x) of x^2 is $2x$, we can say that an indefinite integral of $2x$ is x^2.

In symbols:

$$\frac{d}{dx}(x^2) = 2x, \qquad \text{so} \qquad \int 2x dx = x^2.$$

Note that we say *an* indefinite integral, not *the* indefinite integral. This is because the indefinite integral is not unique. In our example, notice that the derivative of $x^2 + 3$ is also $2x$, so $x^2 + 3$ is another indefinite integral of $2x$. In fact, if c is *any* constant, the derivative of $x^2 + c$ is $2x$ and so $x^2 + c$ is an indefinite integral of $2x$.

We express this in symbols by writing

$$\int 2x dx = x^2 + c$$

where c is what we call an "arbitrary constant". This means that c has no specified value, but can be given any value we like in a particular problem. In this way we encapsulate all possible solutions to the problem of finding an indefinite integral of $2x$ in a single expression.

In most cases, if you are asked to find an indefinite integral of a function, it is not necessary to add the $+c$. However, there are cases in which it is essential. For example, if additional information is given and a specific function has to be found, or if the general solution of a differential equation is sought. (You will learn about these later.) So it is a good idea to get into the habit of adding the arbitrary constant every time, so that when it is really needed you don't forget it.

The inverse relationship between differentiation and integration means that, for every statement about differentiation, we can write down a corresponding statement about integration.

For example,

$$\frac{d}{dx}(x^4) = 4x^3, \qquad \text{so} \qquad \int 4x^3 dx = x^4 + c.$$

Exercises 3.1

Complete the following statements:

(i) $\frac{d}{dx}(\sin x) = \cos x$, so $\int \cos x \, dx = \sin x + c$.

(ii) $\frac{d}{dx}(x^5) =$, so $\int \quad dx =$

(iii) $\frac{d}{dx}(e^x) =$, so $\int \quad dx =$

(iv) $\frac{d}{dx}\left(\frac{1}{x^2}\right) =$, so $\int \quad dx =$

(v) $\frac{d}{dx}(x) =$, so $\int \quad dx =$

(vi) $\frac{d}{dx}(\ln x) =$, so $\int \quad dx =$

The next step is, when we are given a function to integrate, to run quickly through all the standard differentiation formulae in our minds, until we come to one which *fits* our problem.

In other words, we have to learn to recognise a given function as the derivative of another function (where possible).

Try to do the following exercises by *recognising the function* which has the given function as its derivative.

Exercises 3.2

i $\int (-\sin x) dx$

ii $\int 3x^2 dx$

iii $\int 2dx$

iv $\int \sec^2 x dx$

v $\int \frac{3}{2} x^{\frac{1}{2}} dx$

vi $\int \left(-\frac{1}{x^2}\right) dx$

4 Some Rules for Calculating Integrals

Rules for operating with integrals are derived from the rules for operating with derivatives. So, because

$$\frac{d}{dx}\left(cf(x)\right) = c\frac{d}{dx}\left(f(x)\right), \text{ for any constant } c,$$

we have

Rule 1

$$\int (cf(x))dx = c\int f(x)dx, \text{ for any constant } c.$$

For example $\int 10\cos x dx = 10\int \cos x dx = 10\sin x + c.$

It sometimes helps people to understand and remember rules like this if they say them in words. The rule given above says: *The integral of a constant multiple of a function is a constant multiple of the integral of the function.* Another way of putting it is *You can move a constant past the integral sign without changing the value of the expression.*

Similarly, from

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)),$$

we can derive the rule

Rule 2

$$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx.$$

For example,
$$\begin{aligned}\int (e^x + 2x)dx &= \int e^x dx + \int 2x dx \\ &= e^x + x^2 + c.\end{aligned}$$

In words, *the integral of the sum of two functions is the sum of their integrals.*

We can easily extend this rule to include differences as well as sums, and to the case where there are more than two terms in the sum.

Examples

Find the following indefinite integrals:

i $\int (1 + 2x - 3x^2 + \sin x)dx$

ii $\int (3\cos x - \frac{1}{2}e^x)dx$

Solutions

i

$$\begin{aligned}\int (1 + 2x - 3x^2 + \sin x)dx &= \int 1dx + \int 2xdx - \int 3x^2dx - \int (-\sin x)dx \\ &= x + x^2 - x^3 - \cos x + c.\end{aligned}$$

Note: We have written $+\int \sin xdx$ as $-\int(-\sin x)dx$ because $(-\sin x)$ is the derivative of $\cos x$.

ii

$$\begin{aligned}\int (3\cos x - \frac{1}{2}e^x)dx &= 3\int \cos xdx - \frac{1}{2}\int e^x dx \\ &= 3\sin x - \frac{1}{2}e^x + c.\end{aligned}$$

You will find you can usually omit the first step and write the answer immediately.

Exercises 4

Find the following indefinite integrals:

i $\int (\cos x + \sin x)dx$

ii $\int (e^x - 1)dx$

iii $\int (1 - 10x + 9x^2)dx$

iv $\int (3\sec^2 x + \frac{4}{x})dx$

5 Integrating Powers of x and Other Elementary Functions

We can now work out how to integrate any power of x by looking at the corresponding rule for differentiation:

$$\frac{d}{dx}(x^n) = nx^{n-1}, \qquad \text{so} \qquad \int nx^{n-1}dx = x^n + c.$$

Similarly

$$\frac{d}{dx}\left(x^{n+1}\right) = (n+1)x^n, \qquad \text{so} \qquad \int (n+1)x^n dx = x^{n+1} + c.$$

Therefore

$$\begin{aligned} \int x^n dx &= \int \frac{1}{n+1} \cdot (n+1)x^n dx && \leftarrow \text{notice that } \tfrac{1}{n+1} \cdot (n+1) \text{ is just 1 when we cancel} \\ &= \frac{1}{n+1} \int (n+1)x^n dx && \leftarrow \text{take } \tfrac{1}{n+1} \text{ outside the } \textstyle\int \text{ sign} \\ &= \frac{1}{n+1} x^{n+1} + c. \end{aligned}$$

We should now look carefully at the formula we have just worked out and ask: for which values of n does it hold?

Remember that the differentiation rule $\frac{d}{dx}(x^n) = nx^{n-1}$ holds whether n is positive or negative, a whole number or a fraction or even irrational; in other words, for all real numbers n.

We might expect the integration rule to hold for all real numbers also. But there is one snag: in working it out, we divided by $n+1$. Since division by zero does not make sense, the rule will not hold when $n+1 = 0$, that is, when $n = -1$. So we conclude that

Rule 3

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + c$$

for all real numbers n, except $n = -1$.

When $n = -1$, $\int x^n dx$ becomes $\int x^{-1} dx = \int \frac{1}{x} dx$. We don't need to worry that the rule above doesn't apply in this case, because we already know the integral of $\frac{1}{x}$.

Since

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \qquad \text{we have} \qquad \int \frac{1}{x} dx = \ln x + c.$$

Examples

Find

i $\int x^3 dx$

ii $\displaystyle\int \frac{dx}{x^2}$

iii $\int \sqrt{x} dx$

Solutions

i $\displaystyle\int x^3 dx = \frac{1}{(3+1)}x^4 + c = \frac{1}{4}x^4 + c.$ ← replacing n by 3 in the formula

ii $\displaystyle\int \frac{dx}{x^2} = \int x^{-2} dx = \frac{1}{-2+1}x^{-2+1} + c = -\frac{1}{x} + c.$ ← replacing n by -2 in the formula

iii $\displaystyle\int \sqrt{x} dx = \int x^{\frac{1}{2}} dx = \frac{1}{\frac{1}{2}+1}x^{\frac{1}{2}+1} + c = \frac{2}{3}x^{\frac{3}{2}} + c.$ ← replacing n by $\frac{1}{2}$

Exercises 5.1

1. Find anti-derivatives of the following functions:

i x^5 **ii** x^9

iii x^{-4} **iv** $\frac{1}{x^2}$

v $\frac{1}{\sqrt{x}}$ **vi** $\sqrt[3]{x}$

vii $x^{\sqrt{2}}$ **viii** $x\sqrt{x}$

ix $\frac{1}{x^\pi}$

2. Find the following integrals:

i $\displaystyle\int -3x dx$

ii $\displaystyle\int \left(x^3 + 3x^2 + x + 4\right) dx$

iii $\displaystyle\int \left(x - \frac{1}{x}\right) dx$

iv $\displaystyle\int \left(x - \frac{1}{x}\right)^2 dx$ Hint: multiply out the expression

v $\displaystyle\int \left(\frac{2}{\sqrt{x}} + \frac{\sqrt{x}}{2}\right) dx$

vi $\displaystyle\int \frac{2x^4 + x^2}{x} dx$ Hint: divide through by the denominator

vii $\displaystyle\int \left(\frac{3 + 5x - 6x^2 - 7x^3}{2x^2}\right) dx$ Hint: divide through by the denominator

At this stage it is very tempting to give a list of standard integrals, corresponding to the list of derivatives given at the beginning of this booklet. However, you are NOT encouraged to memorise integration formulae, but rather to become VERY familiar with the list of derivatives and to practise recognising a function as the derivative of another function.

If you try memorising *both* differentiation *and* integration formulae, you will one day mix them up and use the wrong one. And there is absolutely *no need* to memorise the integration formulae if you know the differentiation ones.

It is much better to recall the way in which an integral is defined as an anti-derivative. *Every time you perform an integration* you should pause for a moment and check it by differentiating the answer to see if you get back the function you began with. This is a very important habit to develop. There is no need to write down the checking process every time, often you will do it in your head, but if you get into this habit you will avoid a lot of mistakes.

There is a table of derivatives at the front of this booklet. Try to avoid using it if you can, but refer to it if you are unsure.

Examples

Find the following indefinite integrals:

i $\int \left(e^x + 3x^{\frac{5}{2}}\right) dx$

ii $\int (5\csc^2 x + 3\sec^2 x)dx$

Solutions

i

$$\begin{aligned}\int \left(e^x + 3x^{\frac{5}{2}}\right) dx &= \int e^x dx + 3\int x^{\frac{5}{2}} dx \\ &= e^x + 3\cdot\frac{1}{\frac{5}{2}+1}x^{\frac{5}{2}+1} + c \\ &= e^x + 3\cdot\frac{2}{7}x^{\frac{7}{2}} + c \\ &= e^x + \frac{6}{7}x^{\frac{7}{2}} + c.\end{aligned}$$

ii

$$\begin{aligned}\int (5\csc^2 x + 3\sec^2 x)dx &= -5\int(-\csc^2 x)dx + 3\int \sec^2 x dx \\ &= -5\cot x + 3\tan x + c.\end{aligned}$$

Exercises 5.2

Integrate the following functions with respect to x:

i $10e^x - 5\sin x$

ii $\sqrt{x}(x^2 + x + 1)$ Hint: Multiply through by $\sqrt{x}$, and write with fractional exponents.

iii $\dfrac{5}{\sqrt{(1-x^2)}} + \dfrac{1}{\sqrt{x}}$

iv $\dfrac{x^3 + x + 1}{1 + x^2}$ Hint: Divide through by $1+x^2$, and consult table of derivatives.

v $\dfrac{\tan x}{\sin x \cos x}$ Hint: Write $\tan x$ as $\frac{\sin x}{\cos x}$ and simplify.

vi $\tan^2 x$ Hint: Remember the formula $1 + \tan^2 x = \sec^2 x$.

You may use the table of derivatives if you like.

(If you are not familiar with inverse trig functions, omit parts **iii** and **iv**.)

Hint: In order to get some of the functions above into a form in which we can recognise what they are derivatives of, we may have to express them differently. Try to think of ways in which they could be changed that would be helpful.

6 Things You Can't Do With Integrals

It is just as important to be aware of what you *can't* do when integrating, as to know what you *can* do. In this way you will avoid making some serious mistakes.

We mention here two fairly common ones.

1. We know that $\int cf(x)dx = c\int f(x)dx$, where c is a constant. That is, "you are allowed to move a constant past the integral sign". It is often very tempting to try the same thing with a variable, i.e. to equate $\int xf(x)dx$ with $x\int f(x)dx$.

If we check a few special cases, however, it will become clear that this is **not correct**. For example, compare the values of

$$\int x \cdot xdx \qquad \text{and} \qquad x\int xdx.$$

$$\int x \cdot xdx = \int x^2dx = \frac{1}{3}x^3 + c \qquad \text{while} \qquad x\int xdx = x \cdot \frac{1}{2}x^2 = \frac{1}{2}x^3 + c.$$

These expressesions are obviously different.

So the "rule" we tried to invent does not work!

It is unlikely that anybody would try to find $\int x^2dx$ in the way shown above. However, if asked to find $\int x\sin xdx$, one might very easily be tempted to write $x\int \sin xdx = -x\cos x + c$. Although we do not yet know a method for finding $\int x\sin xdx$, we can very easily show that the answer obtained above is wrong. How?

By differentiating the answer, of course!

If we don't get back to $x\sin x$, we must have gone wrong somewhere.

Notice that $x\cos x$ is a product, so we must use the product rule to differentiate it.

$$\begin{aligned}\frac{d}{dx}(-x\cos x + c) &= -x(-\sin x) + (-1)\cos x \\ &= x\sin x - \cos x.\end{aligned}$$

The first term is correct, but the second term shouldn't be there! So the method we used was wrong.

$$\int xf(x)dx \text{ is } \textbf{not} \text{ equal to } x\int f(x)dx.$$

In words,

you **cannot** move a variable past the integral sign.

2. Again, we know that $\int (f(x) + g(x))\, dx = \int f(x)dx + \int g(x)dx$. That is, "the integral of a sum is equal to the sum of the integrals". It may seem reasonable to wonder whether there is a similar rule for products. That is, whether we can equate $\int f(x) \cdot g(x)dx$ with $\int f(x)dx \cdot \int g(x)dx$.

Once again, checking a few special cases will show that this is **not correct**.

Take, for example, $\int x \sin x dx$.

Now

$$\begin{aligned}\int xdx \cdot \int \sin xdx &= \frac{1}{2}x^2 \cdot (-\cos x) + c \\ &= -\frac{1}{2}x^2 \cos x + c.\end{aligned}$$

But

$$\begin{aligned}\frac{d}{dx}\left(-\frac{1}{2}x^2 \cos x + c\right) &= -\frac{1}{2}x^2 \cdot (-\sin x) + (-x)\cos x \\ &= \frac{1}{2}x^2 \sin x - x\cos x.\end{aligned}$$

product rule again!

and this is nothing like the right answer! (Remember, it ought to have been $x \sin x$.)

So we conclude that

$$\int f(x)g(x) \text{ is } \textbf{not} \text{ equal to } \int f(x)dx \cdot \int g(x)dx.$$

In words,

> the integral of the product of two functions is **not** the same as the product of their integrals.

The other important point you should have learned from this section is the value of *checking* any integration by *differentiating the answer*. If you don't get back to what you started with, you know you have gone wrong somewhere, and since differentiation is generally easier than integration, the mistake is likely to be in the integration.

Exercises 6

Explain the mistakes in the following integrations, and *prove* that the answer obtained in each case is wrong, by differentiating the answers given.

i $\int x^2 e^x dx = \frac{1}{3}x^3 e^x + c$

ii $\int \frac{xdx}{\sqrt{(1-x^2)}} = x \int \frac{1}{\sqrt{(1-x^2)}}dx = x \sin^{-1} x + c$

You will learn how to integrate these functions later.

7 Using the Chain Rule in Reverse

Recall that the Chain Rule is used to differentiate composite functions such as $\cos(x^3+1), e^{\frac{1}{2}x^2}, (2x^2+3)^{11}, \ln(3x+1)$. (The Chain Rule is sometimes called the Composite Functions Rule or Function of a Function Rule.)

If we observe carefully the answers we obtain when we use the chain rule, we can learn to recognise when a function has this form, and so discover how to integrate such functions.

Remember that, if $y = f(u)$ and $u = g(x)$

so that $\qquad y = f(g(x))$, (a composite function)

then $\qquad \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}.$

Using function notation, this can be written as

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In this expression, $f'(g(x))$ is another way of writing $\dfrac{dy}{du}$ where $y = f(u)$ and $u = g(x)$ and $g'(x)$ is another way of writing $\dfrac{du}{dx}$ where $u = g(x)$.

This last form is the one you should learn to recognise.

Examples

By differentiating the following functions, write down the corresponding statement for integration.

i $\sin 3x$

ii $(2x+1)^7$

iii e^{x^2}

Solution

i $\dfrac{d}{dx}\sin 3x = \cos 3x \cdot 3$, so $\int \cos 3x \cdot 3dx = \sin 3x + c$.

ii $\dfrac{d}{dx}(2x+1)^7 = 7(2x+1)^6 \cdot 2$, so $\int 7(2x+1)^6 \cdot 2dx = (2x+1)^7 + c$.

iii $\dfrac{d}{dx}\left(e^{x^2}\right) = e^{x^2} \cdot 2x$, so $\int e^{x^2} \cdot 2xdx = e^{x^2} + c$.

Exercises 7.1

Differentiate each of the following functions, and then rewrite each result in the form of a statement about integration.

i	$(2x-4)^{13}$	**ii**	$\sin \pi x$	**iii**	e^{3x-5}
iv	$\ln(2x-1)$	**v**	$\dfrac{1}{5x-3}$	**vi**	$\tan 5x$
vii	$(x^5-1)^4$	**viii**	$\sin(x^3)$	**ix**	$e^{\sqrt{x}}$
x	$\cos^5 x$	**xi**	$\tan(x^2+1)$	**xii**	$\ln(\sin x)$

The next step is to learn to *recognise* when a function has the forms $f'(g(x)) \cdot g'(x)$, that is, when it is the derivative of a composite function. Look back at each of the integration statements above. In every case, the function being integrated is the *product* of two functions: one is a composite function, and the other is the derivative of the "inner function" in the composite. You can think of it as "the derivative of what's inside the brackets". Note that in some cases, this derivative is a constant.

For example, consider

$$\int e^{3x} \cdot 3dx.$$

We can write e^{3x} as a composite function.
3 is the derivative of $3x$ i.e. the derivative of "what's inside the brackets" in $e^{(3x)}$.

This is in the form

$$\int f'(g(x)) \cdot g'(x)dx$$

with

$$u = g(x) = 3x, \text{ and } f'(u) = e^u.$$

Using the chain rule in reverse, since $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$ we have

$$\int f'(g(x)) \cdot g'(x)dx = f(g(x)) + c.$$

In this case

$$\int e^{3x} \cdot 3dx = e^{3x} + c.$$

If you have any doubts about this, it is easy to check if you are right: *differentiate your answer!*

Now let's try another:

$$\int \cos(x^2+5) \cdot 2xdx.$$

$\cos(x^2+5)$ is a composite function.
$2x$ is the derivative of x^2+5, i.e. the derivative of "what's inside the brackets".

So this is in the form

$$\int f'(g(x)) \cdot g'(x)dx \quad \text{with } u = g(x) = x^2 + 5 \text{ and } f'(u) = \cos u.$$

Recall that if $f'(u) = \cos u$, $f(u) = \sin u$.

So,

$$\int \cos(x^2 + 5) \cdot 2xdx = \sin(x^2 + 5) + c.$$

Again, *check that this is correct*, by differentiating.

People sometimes ask "Where did the $2x$ go?". The answer is, "Back where it came from."

If we differentiate $\sin(x^2 + 5)$ we get $\cos(x^2 + 5) \cdot 2x$.

So when we integrate $\cos(x^2 + 5) \cdot 2x$ we get $\sin(x^2 + 5)$.

Examples

Each of the following functions is in the form $f'(g(x)) \cdot g'(x)$.

Identify $f'(u)$ and $u = g(x)$ and hence find an indefinite integral of the function.

i $(3x^2 - 1)^4 \cdot 6x$

ii $\sin(\sqrt{x}) \cdot \dfrac{1}{2\sqrt{x}}$

Solutions

i $(3x^2 - 1)^4 \cdot 6x$ is a product of $(3x^2 - 1)^4$ and $6x$.

Clearly $(3x^2 - 1)^4$ is the composite function $f'(g(x))$. So $g(x)$ should be $3x^2 - 1$.

$6x$ is the "other part". This should be the derivative of "what's inside the brackets" i.e. $3x^2 - 1$, and clearly, this is the case:

$\dfrac{d}{dx}(3x^2 - 1) = 6x.$

So, $u = g(x) = 3x^2 - 1$ and $f'(u) = u^4$ giving $f'(g(x)) \cdot g'(x) = (3x^2 - 1)^4 \cdot 6x$.

If $f'(u) = u^4$, $f(u) = \frac{1}{5}u^5$.

So, using the rule

$$\int f'(g(x)) \cdot g'(x)dx = f(g(x)) + c$$

we conclude

$$\int (3x^3 - 1)^4 \cdot 6x = \frac{1}{5}(3x^2 - 1)^5 + c.$$

You should differentiate this answer *immediately* and check that you get back the function you began with.

ii $\sin(\sqrt{x}) \cdot \dfrac{1}{2\sqrt{x}}$

This is a product of $\sin(\sqrt{x})$ and $\frac{1}{2\sqrt{x}}$.

Clearly $\sin(\sqrt{x})$ is a composite function.

The part "inside the brackets" is $\sqrt{x}$, so we would like this to be $g(x)$. The other factor $\frac{1}{2\sqrt{x}}$ ought to be $g'(x)$. Let's check if this is the case:

$g(x) = \sqrt{x} = x^{\frac{1}{2}}$, so $g'(x) = \dfrac{1}{2}x^{-\frac{1}{2}} = \dfrac{1}{2x^{\frac{1}{2}}} = \dfrac{1}{2\sqrt{x}}$.

So we're right! Thus $u = g(x) = \sqrt{x}$ and $f'(u) = \sin u$ giving

$$f'(g(x)) \cdot g'(x) = \sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}.$$

Now, if $f'(u) = \sin u$, $f(u) = -\cos u$.

So using the rule

$$\int f'(g(x)) \cdot g'(x) dx = f(g(x)) + c$$

we conclude

$$\int \sin(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} dx = -\cos(\sqrt{x}) + c.$$

Again, *check immediately* by differentiating the answer.

Note: The explanations given here are fairly lengthy, to help you to understand what we're doing. Once you have grasped the idea, you will be able to do these very quickly, without needing to write down any explanation.

Example

Integrate $\displaystyle\int \sin^3 x \cdot \cos x dx$.

Solution

$$\int \sin^3 x \cdot \cos x dx = \int (\sin x)^3 \cdot \cos x dx.$$

So $u = g(x) = \sin x$ with $g'(x) = \cos x$.

And $f'(u) = u^3$ giving $f(u) = \frac{1}{4}u^4$.

Hence $\displaystyle\int \sin^3 x \cdot \cos x dx = \frac{1}{4}(\sin x)^4 + c = \frac{1}{4}\sin^4 x + c$.

Exercises 7.2

Each of the following functions is in the form $f'(g(x)) \cdot g'(x)$. Identify $f'(u)$ and $u = g(x)$ and hence find an indefinite integral of the function.

i	$\dfrac{1}{3x-1} \cdot 3$	**ii**	$\sqrt{2x+1} \cdot 2$	**iii**	$(\ln x)^2 \cdot \dfrac{1}{x}$
iv	$e^{2x+4} \cdot 2$	**v**	$\sin(x^3) \cdot 3x^2$	**vi**	$\cos\left(\dfrac{\pi x}{2}\right) \cdot \dfrac{\pi}{2}$
vii	$(7x-8)^{12} \cdot 7$	**viii**	$\sin(\ln x) \cdot \dfrac{1}{x}$	**ix**	$\left(\dfrac{1}{\sin x}\right) \cdot \cos x$
x	$e^{\tan x} \cdot \sec^2 x$	**xi**	$e^{x^3} \cdot 3x^2$	**xii**	$\sec^2(5x-3) \cdot 5$
xiii	$(2x-1)^{\frac{1}{3}} \cdot 2$	**xiv**	$\sqrt{\sin x} \cdot \cos x$		

The final step in learning to use this process is to be able to recognise when a function is not quite in the correct form but *can be put into the correct form* by minor changes.

For example, we try to calculate $\int x^3\sqrt{x^4+1}dx$.

We notice that $\sqrt{x^4+1}$ is a composite function, so we would like to have $u = g(x) = x^4+1$. But this would mean $g'(x) = 4x^3$, and the integrand (i.e. the function we are trying to integrate) only has x^3. However, we can easily make it $4x^3$, as follows:

$$\int x^3\sqrt{x^4+1}dx = \frac{1}{4}\int \sqrt{x^4+1} \cdot 4x^3dx.$$

Note: The $\frac{1}{4}$ and the 4 cancel with each other, so the expression is not changed.

So $u = g(x) = x^4 + 1$, $\qquad g'(x) = 4x^3$

And $f'(u) = u^{\frac{1}{2}}$ $\qquad f(u) = \frac{2}{3}u^{\frac{3}{2}}$

So, $\displaystyle\int x^3\sqrt{x^4+1}dx = \frac{1}{4}\int \sqrt{x^4+1} \cdot 4x^2dx = \frac{1}{4} \cdot \frac{2}{3}\left(x^4+1\right)^{\frac{3}{2}} + c.$

Note: We may only insert constants in this way, not variables.

We *cannot* for example evaluate $\displaystyle\int e^{x^2}dx$ by writing $\displaystyle\frac{1}{2x}\int e^{x^2} \cdot 2xdx$, because the $\frac{1}{x}$ in front of the integral sign does *n*ot cancel with the x which has been inserted in the integrand.

This integral cannot, in fact, be evaluated in terms of elementary functions.

The example above illustrates one of the difficulties with integration: many seemingly simple functions cannot be integrated without inventing new functions to express the integrals. There is no set of rules which we can apply which will tell us how to integrate any function. All we can do is give some techniques which will work for *some* functions.

Exercises 7.3

Write the following functions in the form $f'(g(x)) \cdot g'(x)$ and hence integrate them:

i	$\cos 7x$	**ii**	xe^{x^2}
iii	$\frac{x}{1-2x^2}$	**iv**	$x^2(4x^3+3)^9$
v	$\sin(1+3x)$	**vi**	$\frac{\sin\sqrt{x}}{\sqrt{x}}$
vii	$\frac{x}{\sqrt{(1-x^2)}}$	**viii**	e^{3x}
ix	$\tan 6x$		

Hint: Write $\tan 6x$ in terms of $\sin 6x$ and $\cos 6x$.

8 Applications

Applications of anti-differentiation arise in problems in which we know the rate of change of a function and want to find the function itself. Problems about motion provide many examples, such as those in which the velocity of a moving object is given and we want to find its position at any time. Since velocity is *rate of change of displacement*, we must anti-differentiate to find the displacement.

Examples

1. A stone is thrown upwards from the top of a tower 50 metres high. Its velocity in an upwards direction t second later is $20 - 5t$ metres per second. Find the height of the stone above the ground 3 seconds later.

Solution

Let the height of the stone above ground level at time t be h metres.

We are given two pieces of information in this problem:

i the fact that the tower is 50m high tells us that when $t = 0$ (that is, at the moment the stone leaves the thrower's hand), $h = 50$,

ii the fact that the velocity at time t is $20 - 5t$ tells us that

$$\frac{dh}{dt} = 20 - 5t.$$

We begin with the second statement, which tells us about *rate of change* of a function. By anti-differentiating, we obtain

$$\begin{aligned} h &= \int (20 - 5t)dt \\ &= 20t - \frac{5}{2}t^2 + c. \qquad (1) \end{aligned}$$

Note: it is vitally important not to forget the '$+c$' (the constant of integration) in problems like this.

Now we can make use of the first statement, which is called an *initial condition* (it tells us what things were like at the start) to find a value for the constant c. By substituting $h = 50$ and $t = 0$ into the equation (1), we obtain

$$50 = c.$$

$$\text{So } h = 20t - \tfrac{5}{2}t^2 + 50. \qquad (2)$$

Finally, let us go back to the problem, and read it again, to check exactly what we are asked to find: 'Find the height of the stone above the ground 3 seconds later'. To find this, all we have to do is substitute $t = 3$ in the expression we have just derived.

When $t = 3$, $h = 60 - \frac{5}{2} \cdot 9 + 50 = 87.5$, so the height of the stone 3 seconds later is 87.5 metres.

Let us look back at the structure of this problem and its solution.

- We are given information about the *rate of change* of a quantity, and we anti-differentiate (i.e. integrate) to get a general expression for the quantity, including an arbitary constant.
- We are given information about initial conditions and use this to find the value of the constant of integration.
- We now have a precise expression for the quantity, and can use that to answer the questions asked.

2. When a tap at the base of a storage tank is turned on, water flows out of the tank at the rate of $200e^{-\frac{1}{5}t}$ litres per minute. If the volume of water in the tank at the start is 1000 litres, find how much is left after the tap has been running for 10 minutes.

Solution

Let the volume of water in the tank t minutes after the tap is turned on be V litres. Since water is running *out* of the tank, V will be *decreasing*, and so $\frac{dV}{dt}$ must be negative.

$$
\begin{array}{llcl}
\text{Hence} & \dfrac{dV}{dt} & = & -200e^{-\frac{1}{5}t} \\
\text{and so} & V & = & \displaystyle\int \left(-200e^{-\frac{1}{5}t}\right) dt \\
 & & = & -200 \cdot (-5) \displaystyle\int e^{-\frac{1}{5}t} \left(-\frac{1}{5}\right) dt \\
 & & = & 1000e^{-\frac{1}{5}t} + c.
\end{array}
$$

Now when $t = 0$, $V = 1000$, so $1000 = 1000 + c$, hence $c = 0$.

So at any time $t, V = 1000e^{-\frac{1}{5}t}$

Thus when $t = 10$, $V = 1000e^{-2}$
≈ 135.34 litres.

Exercises 8

1. When a stone is dropped into smooth water, circular ripples spread out from the point where it enters the water. If the area covered by ripples increases at a rate of $2\pi t$ square metres per second, find the total area of disturbed water t seconds after the stone hits the water. What is the area covered by ripples after 3 seconds?

[Note: when $t = 0$, the stone is just entering the water, so the area of disturbed water is 0.]

2. A population of animals, under certain conditions, has a growth rate given by $500\pi\cos 2\pi t$ where t is the time in years. If the initial size of the herd is 3000, find the size of the population at time t. What are the maximum and minimum numbers in the herd during the course of a year?

3. During an experiment, the height of a growing plant increased at a rate of $\frac{1}{\sqrt{t+4}}$ cm per day, where t represents the number of days since the start of the experiment. If the plant was 20 cm high at the beginning, what would its height be after 12 days?

4. An oral dose of a drug was administered to a patient. t hours later, the concentration of the drug in the patient's blood was changing at a rate give by $5e^{-t} - e^{-0.2t}$. If none of the drug was present in the blood at the time the dose was given, find the concentration of the drug in the patient's blood t hours later.

How long after administration will the concentration be greatest?

5. An object is propelled along the ground in such a way that its velocity after t seconds is $\frac{1}{t+1}$ metres per second. If it starts 2 metres from a fixed point, and moves in a straight line directly towards the point, how long will it take to reach the point?

9 Solutions to Exercises

Exercises 3.1

i $\frac{d}{dx}(\sin x) = \cos x$, so $\int \cos x dx = \sin x + c.$

ii $\frac{d}{dx}(x^5) = 5x^4$, so $\int 5x^4 dx = x^5 + c.$

iii $\frac{d}{dx}(e^x) = e^x$, so $\int e^x dx = e^x + c.$

iv $\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}$, so $\int -\frac{2}{x^3} dx = \frac{1}{x^2} + c.$

$\left(\frac{1}{x^2} = x^{-2} \text{ and } \frac{d}{dx}(x^{-2}) = -2x^{-3} = -\frac{2}{x^3}\right)$

v $\frac{d}{dx}(x) = 1$, so $\int 1 dx = x + c.$

(**Note:** $\int 1dx$ is usually written as $\int dx$.)

vi $\frac{d}{dx}(\ln x) = \frac{1}{x}$, so $\int \frac{1}{x} dx = \ln x + c.$

Exercises 3.2

Note: All these answers can be checked by differentiating!

$$
\begin{array}{lll}
\textbf{i} & \displaystyle\int (-\sin x)dx & = \cos x + c. \\
\textbf{ii} & \displaystyle\int 3x^2 dx & = x^3 + c. \\
\textbf{iii} & \displaystyle\int 2dx & = 2x + c. \\
\textbf{iv} & \displaystyle\int \sec^2 x dx & = \tan x + c. \\
\textbf{v} & \displaystyle\int \frac{3}{2} x^{\frac{1}{2}} dx & = x^{\frac{3}{2}} + c. \\
\textbf{vi} & \displaystyle\int -\frac{1}{x^2} dx & = \frac{1}{x} + c.
\end{array}
$$

Exercises 4

$$
\begin{array}{lll}
\textbf{i} & \displaystyle\int (\cos x + \sin x)dx & = \sin x - \cos x + c. \\
\textbf{ii} & \displaystyle\int (e^x - 1)dx & = e^x - x + c. \\
\textbf{iii} & \displaystyle\int (1 - 10x + 9x^2)dx & = x - 5x^2 + 3x^3 + c. \\
\textbf{iv} & \displaystyle\int (3\sec^2 x + \frac{4}{x})dx & = 3\tan x + 4\ln x + c.
\end{array}
$$

Exercises 5.1

1. **i** $\int x^5 dx = \frac{1}{6}x^6 + c.$

ii $\int x^9 dx = \frac{1}{10}x^{10} + c.$

iii $\int x^{-4} dx = -\frac{1}{3}x^{-4+1} = -\frac{1}{3}x^{-3} + c = -\frac{1}{3x^3} + c.$

iv $\int \frac{1}{x^2} dx = \int x^{-2} dx = \frac{1}{-1}x^{-1} + c = -\frac{1}{x} + c.$

v $\int \frac{1}{\sqrt{x}} dx = \int x^{-\frac{1}{2}} dx = 2x^{\frac{1}{2}} + c = 2\sqrt{x} + c.$

vi $\int \sqrt[3]{x} dx = \int x^{\frac{1}{3}} dx = \frac{3}{4}x^{\frac{1}{3}+1} = \frac{3}{4}x^{\frac{4}{3}} + c.$

(**Note:** In exercises like **v** and **vi** above, it is easier to work out what power of x is required, and then to work out what coefficient is needed to give the correct answer on differentiating. This is usually better than substituting for n in $\frac{1}{n+1}x^{n+1}$. So **v** is more easily done by saying (mentally) "$-\frac{1}{2} + 1 = \frac{1}{2}$, so the answer will involve $x^{\frac{1}{2}}$. Now $2x^{\frac{1}{2}}$ will give a coefficient of 1 when differentiated so $\int x^{-\frac{1}{2}} dx = 2x^{\frac{1}{2}} + c$ ").

vii $\int x^{\sqrt{2}} dx = \frac{1}{\sqrt{2}+1}x^{\sqrt{2}+1} + c.$

viii $\int x\sqrt{x} dx = \int x^{\frac{3}{2}} dx = \frac{2}{5}x^{\frac{5}{2}} + c = \frac{2}{5}x^2\sqrt{x} + c.$

ix $\int \frac{1}{x^\pi} dx = \int x^{-\pi} dx = \frac{1}{-\pi+1}x^{-\pi+1} + c = -\frac{1}{(\pi-1)x^{\pi-1}} + c.$

2. **i** $\int -3x dx = -3 \cdot \frac{1}{2}x^2 + c = -\frac{3}{2}x^2 + c.$

ii $\int (x^3 + 3x^2 + x + 4) dx = \frac{1}{4}x^4 + x^3 + \frac{1}{2}x^2 + 4x + c.$

iii $\int (x - \frac{1}{x}) dx = \frac{1}{2}x^2 - \ln x + c.$

iv
$$\begin{aligned}\int (x - \frac{1}{x})^2 dx &= \int (x^2 - 2 + \frac{1}{x^2}) dx \\ &= \frac{1}{3}x^3 - 2x - \frac{1}{x} + c. \qquad \text{(Recall, } \int \frac{1}{x^2} dx = -\frac{1}{x}\text{)}\end{aligned}$$

v $\int \left(\frac{2}{\sqrt{x}} + \frac{\sqrt{x}}{2} \right) dx = 2\int x^{-\frac{1}{2}} dx + \frac{1}{2}\int x^{\frac{1}{2}} dx$

$$= \quad 2 \cdot 2x^{\frac{1}{2}} + \frac{1}{2} \cdot \frac{2}{3} x^{\frac{3}{2}} + c$$

$$= \quad 4\sqrt{x} + \frac{1}{3} x\sqrt{x} + c.$$

vi $\displaystyle\int \frac{2x^4 + x^2}{x} dx = \int (2x^3 + x) dx$

$$= \quad 2 \cdot \frac{1}{4} x^4 + \frac{1}{2} x^2 + c$$

$$= \quad \frac{1}{2} x^4 + \frac{1}{2} x^2 + c.$$

vii $\displaystyle\int \frac{3 + 5x - 6x^2 - 7x^3}{2x^2} dx = \int \left(\frac{3}{2x^2} + \frac{5}{2x} - 3 - \frac{7x}{2} \right) dx$

$$= \quad \frac{3}{2} \int x^{-2} dx + \frac{5}{2} \int \frac{1}{x} dx - 3 \int dx - \frac{7}{2} \int x dx$$

$$= \quad \frac{3}{2}(-\frac{1}{x}) + \frac{5}{2} \ln x - 3x - \frac{7}{2} \cdot \frac{1}{2} x^2 + c$$

$$= \quad -\frac{3}{2x} + \frac{5}{2} \ln x - 3x - \frac{7}{4} x^2 + c.$$

Exercises 5.2

i $\displaystyle\int (10e^x - 5\sin x) dx = 10e^x + 5\cos x + c.$

ii $\displaystyle\int \sqrt{x}(x^2 + x + 1) dx = \int \left(x^{\frac{5}{2}} + x^{\frac{3}{2}} + x^{\frac{1}{2}} \right) dx$

$$= \quad \frac{2}{7} x^{\frac{7}{2}} + \frac{2}{5} x^{\frac{5}{2}} + \frac{2}{3} x^{\frac{3}{2}} + c.$$

iii $\displaystyle\int \left(\frac{5}{\sqrt{(1 - x^2)}} + \frac{1}{\sqrt{x}} \right) dx = 5\sin^{-1} x + 2\sqrt{x} + c.$

iv $\displaystyle\int \frac{x^3 + x + 1}{1 + x^2} dx = \int \frac{x(x^2 + 1) + 1}{1 + x^2} dx$

$$= \quad \int \left(x + \frac{1}{1 + x^2} \right) dx$$ Dividing through by $(1 + x^2)$

$$= \quad \frac{1}{2} x^2 + \tan^{-1} x + c.$$

v $\displaystyle\int \frac{\tan x}{\sin x \cos x} dx = \int \frac{\sin x}{\cos x} \cdot \frac{1}{\sin x \cos x} dx$ Writing $\tan x = \frac{\sin x}{\cos x}$

$$\begin{aligned}
&= \int \frac{1}{\cos^2 x} dx \\
&= \int \sec^2 x dx \\
&= \tan x + c.
\end{aligned}$$

vi $\displaystyle\int \tan^2 x dx = \int (\sec^2 x - 1) dx$ Using $\tan^2 x = \sec^2 x - 1$

$$\begin{aligned}
&= \int \sec^2 x dx - \int 1 dx \\
&= \tan x - x + c.
\end{aligned}$$

Exercises 6

i The mistake arises from writing the integral of a product as the product of the integrals of the factors. We **prove** the answer obtained is wrong by differentiating it, and showing that we do not get back the function we were trying to integrate.

$$\underbrace{\tfrac{d}{dx}\left(\tfrac{1}{3}x^3e^x\right)}_{\mathbf{\frac{d}{dx}(uv)}} = \underbrace{\tfrac{1}{3}x^3e^x}_{\mathbf{u\frac{dv}{dx}}} + \underbrace{x^2e^x}_{\mathbf{\frac{du}{dx}v}}$$

← NOT the same as x^2e^x because of the extra term!

ii The mistake arises from taking the x outside the integral sign. We cannot do this because x is a variable. Again, we **prove** the answer wrong by differentiating.

$$\underbrace{\tfrac{d}{dx}(x\sin^{-1} x)}_{\mathbf{\frac{d}{dx}(uv)}} = \underbrace{x \cdot \tfrac{1}{\sqrt{(1-x^2)}}}_{\mathbf{u\frac{dv}{dx}}} + \underbrace{1 \cdot \sin^{-1} x}_{\mathbf{\frac{du}{dx}v}}$$

$$= \tfrac{x}{\sqrt{(1-x^2)}} + \sin^{-1} x$$

Again, this is **not** the same as $\frac{x}{\sqrt{(1-x^2)}}$ because of the extra term!

Exercises 7.1

i $\dfrac{d}{dx}(2x-4)^{13} = 13 \cdot (2x-4)^{12} \cdot 2,$ so $\displaystyle\int 13(2x-4)^{12} \cdot 2dx = (2x-4)^{13} + c.$

ii $\dfrac{d}{dx}(\sin \pi x) = \cos \pi x \cdot \pi,$ so $\displaystyle\int \cos \pi x \cdot \pi dx = \sin \pi x + c.$

iii $\dfrac{d}{dx}(e^{3x-5}) = e^{3x-5} \cdot 3,$ so $\displaystyle\int e^{3x-5} \cdot 3dx = e^{3x-5} + c.$

iv $\dfrac{d}{dx}(\ln(2x-1)) = \dfrac{1}{2x-1} \cdot 2,$ so $\displaystyle\int \frac{1}{2x-1} \cdot 2dx = \ln(2x-1) + c.$

v $\dfrac{d}{dx}(\dfrac{1}{5x-3}) = -\dfrac{1}{(5x-3)^2} \cdot 5,$ so $\displaystyle\int -\frac{1}{(5x-3)^2} \cdot 5dx = \frac{1}{5x-3} + c.$

$$\begin{array}{llllll}
\textbf{vi} & \dfrac{d}{dx}(\tan 5x) & = \sec^2 5x \cdot 5, & \text{so} \displaystyle\int \sec^2 5x \cdot 5dx & = & \tan 5x + c. \\
\textbf{vii} & \dfrac{d}{dx}((x^5-1)^4) & = 4(x^5-1)^3 \cdot 5x^4, & \text{so} \displaystyle\int 4(x^5-1)^3 \cdot 5x^4 dx & = & (x^5-1)^4 + c. \\
\textbf{viii} & \dfrac{d}{dx}(\sin x^3) & = \cos(x^3) \cdot 3x^2, & \text{so} \displaystyle\int \cos(x^3) \cdot 3x^2 dx & = & \sin(x^3) + c. \\
\textbf{ix} & \dfrac{d}{dx}(e^{\sqrt{x}}) & = e^{\sqrt{x}} \cdot \dfrac{1}{2}x^{-\frac{1}{2}}, & \text{so} \displaystyle\int e^{\sqrt{x}} \cdot \dfrac{1}{2}x^{-\frac{1}{2}} dx & = & e^{\sqrt{x}} + c. \\
\textbf{x} & \dfrac{d}{dx}(\cos^5 x) & = 5\cos^4 x \cdot (-\sin x), & \text{so} \displaystyle\int 5\cos^4 x \cdot (-\sin x) dx & = & \cos^5 x + c. \\
\textbf{xi} & \dfrac{d}{dx}(\tan(x^2+1)) & = \sec^2(x^2+1) \cdot 2x, & \text{so} \displaystyle\int \sec^2(x^2+1) \cdot 2x dx & = & \tan(x^2+1) + c. \\
\textbf{xii} & \dfrac{d}{dx}(\ln(\sin x)) & = \dfrac{1}{\sin x} \cdot \cos x, & \text{so} \displaystyle\int \dfrac{1}{\sin x} \cdot \cos x dx & = & \ln(\sin x) + c.
\end{array}$$

Exercises 7.2

(Before you read these solutions, *check* your work by differentiating your answer.)

i $\displaystyle\int \frac{1}{3x-1} \cdot 3dx = \ln(3x-1) + c.$

$$\left\{ \begin{array}{llll} u = g(x) & = 3x-1 & \text{so } g'(x) & = 3 \\ f'(u) & = \frac{1}{u} & \text{so } f(u) & = \ln u \end{array} \right.$$

ii $\displaystyle\int \sqrt{2x+1} \cdot 2dx = \frac{2}{3}(2x+1)^{\frac{3}{2}} + c.$

$$\left\{ \begin{array}{llll} u = g(x) & = 2x+1 & \text{so } g'(x) & = 2 \\ f'(u) & = \sqrt{u} & \text{so } f(u) & = \frac{2}{3}u^{\frac{3}{2}} \end{array} \right.$$

iii $\displaystyle\int (\ln x)^2 \cdot \frac{1}{x} dx = \frac{1}{3}(\ln x)^3 + c.$

$$\left\{ \begin{array}{llll} u = g(x) & = \ln x & \text{so } g'(x) & = \frac{1}{x} \\ f'(u) & = u^2 & \text{so } f(u) & = \frac{1}{3}u^3 \end{array} \right.$$

iv $\displaystyle\int e^{2x+4} \cdot 2dx = e^{2x+4} + c.$

$$\left\{ \begin{array}{llll} u = g(x) & = 2x+4 & \text{so } g'(x) & = 2 \\ f'(u) & = e^u & \text{so } f(u) & = e^u \end{array} \right.$$

v $\int \sin(x^3) \cdot 3x^2 dx = -\cos(x^3) + c.$

$$\begin{cases} u = g(x) & = x^3 & \text{so } g'(x) & = 3x^2 \\ f'(u) & = \sin u & \text{so } f(u) & = -\cos u \end{cases}$$

vi $\int \cos(\frac{\pi x}{2}) \cdot \frac{\pi}{2} dx = \sin(\frac{\pi x}{2}) + c.$

$$\begin{cases} u = g(x) & = \frac{\pi}{2}x & \text{so } g'(x) & = \frac{\pi}{2} \\ f'(u) & = \cos u & \text{so } f(u) & = \sin u \end{cases}$$

vii $\int (7x-8)^{12} \cdot 7dx = \frac{1}{13}(7x-8)^{13} + c.$

$$\begin{cases} u = g(x) & = 7x - 8 & \text{so } g'(x) & = 7 \\ f'(u) & = u^{12} & \text{so } f(u) & = \frac{1}{13}u^{13} \end{cases}$$

viii $\int \sin(\ln x) \cdot \frac{1}{x} dx = -\cos(\ln x) + c.$

$$\begin{cases} u = g(x) & = \ln x & \text{so } g'(x) & = \frac{1}{x} \\ f'(u) & = \sin u & \text{so } f(u) & = -\cos u \end{cases}$$

ix $\int \frac{1}{\sin x} \cdot \cos x dx = \ln(\sin x) + c.$

$$\begin{cases} u = g(x) & = \sin x & \text{so } g'(x) & = \cos x \\ f'(u) & = \frac{1}{u} & \text{so } f(u) & = \ln u \end{cases}$$

x $\int e^{\tan x} \cdot \sec^2 x dx = e^{\tan x} + c.$

$$\begin{cases} u = g(x) & = \tan x & \text{so } g'(x) & = sec^2 x \\ f'(u) & = e^u & \text{so } f(u) & = e^u \end{cases}$$

xi $\int e^{x^3} \cdot 3x^2 dx = e^{x^3} + c.$

$$\begin{cases} u = g(x) & = x^3 & \text{so } g'(x) & = 3x^2 \\ f'(u) & = e^u & \text{so } f(u) & = e^u \end{cases}$$

xii $\int \sec^2(5x-3) \cdot 5dx = \tan(5x-3) + c.$

$$\begin{cases} u = g(x) & = 5x - 3 & \text{so } g'(x) & = 5 \\ f'(u) & = \sec^2 u & \text{so } f(u) & = \tan u \end{cases}$$

xiii $\int (2x-1)^{\frac{1}{3}} \cdot 2dx = \frac{3}{4}(2x-1)^{\frac{4}{3}} + c.$

$$\left\{\begin{array}{llll} u = g(x) & = 2x - 1 & \text{so } g'(x) & = 2 \\ f'(u) & = u^{\frac{1}{3}} & \text{so } f(u) & = \frac{3}{4}u^{\frac{4}{3}} \end{array}\right.$$

xiv $\displaystyle\int \sqrt{\sin x} \cdot \cos x dx = \frac{2}{3}(\sin x)^{\frac{3}{2}} + c.$

$$\left\{\begin{array}{llll} u = g(x) & = \sin x & \text{so } g'(x) & = \cos x \\ f'(u) & = \sqrt{u} & \text{so } f(u) & = \frac{2}{3}u^{\frac{3}{2}} \end{array}\right.$$

Exercises 7.3

(Before reading the solutions, check all your answers by differentiating!)

i $\displaystyle\int \cos 7x dx = \frac{1}{7}\int \cos 7x \cdot 7dx = \frac{1}{7}\sin 7x + c.$

$$\left\{\begin{array}{rcl} u & = & g(x) = 7x,\;\; g'(x) = 7 \\ f'(u) & = & \cos u \textbf{ so } f(u) = \sin u \end{array}\right.$$

ii $\displaystyle\int xe^{x^2} dx = \frac{1}{2}\int e^{x^2} \cdot 2xdx = \frac{1}{2}e^{x^2} + c.$

$$\left\{\begin{array}{rcl} u & = & g(x) = x^2,\;\; g'(x) = 2x \\ f'(u) & = & e^u \textbf{ so } f(u) = e^u \end{array}\right.$$

iii $\displaystyle\int \frac{x}{1-2x^2}dx = -\frac{1}{4}\int \frac{1}{1-2x^2} \cdot (-4x)dx = -\frac{1}{4}\ln(1-2x^2) + c.$

$$\left\{\begin{array}{rcl} u & = & g(x) = 1-2x^2,\;\; g'(x) = -4x \\ f'(u) & = & \frac{1}{u} \textbf{ so } f(u) = \ln u \end{array}\right.$$

iv $\displaystyle\int x^2(4x^3+3)^9 dx = \frac{1}{12}\int (4x^3+3)^9 \cdot 12x^2 dx = \frac{1}{12}\cdot\frac{1}{10}(4x^3+3)^{10} + c = \frac{1}{120}(4x^3+3)^{10} + c.$

$$\left\{\begin{array}{rcl} u & = & g(x) = 4x^3+3,\;\; g'(x) = 12x^2 \\ f'(u) & = & u^9 \textbf{ so } f(u) = \frac{1}{10}u^{10} \end{array}\right.$$

v $\displaystyle\int \sin(1+3x)dx = \frac{1}{3}\int \sin(1+3x)\cdot 3dx = -\frac{1}{3}\cos(1+3x) + c.$

$$\left\{\begin{array}{rcl} u & = & g(x) = 1+3x,\;\; g'(x) = 3 \\ f'(u) & = & \sin u \textbf{ so } f(u) = -\cos u \end{array}\right.$$

vi $\displaystyle\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = 2\int \sin\sqrt{x}\cdot\frac{1}{2\sqrt{x}}dx = -2\cos\sqrt{x} + c.$

$$\left\{\begin{array}{rcl} u & = & g(x) = \sqrt{x},\;\; g'(x) = \frac{1}{2\sqrt{x}} \\ f'(u) & = & \sin u \textbf{ so } f(u) = -\cos u \end{array}\right.$$

vii $\displaystyle\int \frac{x}{\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{1}{\sqrt{1-x^2}} \cdot (-2x)dx = -\frac{1}{2} \cdot 2(1-x^2)^{\frac{1}{2}} + c = -(1-x^2)^{\frac{1}{2}} + c.$

$$\left\{\begin{array}{rcl} u &=& g(x) = 1-x^2,\ \ g'(x) = -2x \\ f'(u) &=& \frac{1}{\sqrt{u}} \textbf{ so } f(u) = 2u^{\frac{1}{2}} \end{array}\right.$$

viii $\displaystyle\int e^{3x}dx = \frac{1}{3}\int e^{3x} \cdot 3dx = \frac{1}{3}e^{3x} + c.$

$$\left\{\begin{array}{rcl} u &=& g(x) = 3x,\ \ g'(x) = 3 \\ f'(u) &=& e^u \textbf{ so } f(u) = e^u \end{array}\right.$$

ix $\displaystyle\int \tan 6x dx = \int \frac{\sin 6x}{\cos 6x}dx = -\frac{1}{6}\int \frac{1}{\cos 6x} \cdot -6\sin 6x = -\frac{1}{6}\ln(\cos 6x) + c.$

$$\left\{\begin{array}{rcl} u &=& g(x) = \cos 6x,\ \ g'(x) = -6\sin 6x \\ f'(u) &=& \frac{1}{u} \textbf{ so } f(u) = \ln u \end{array}\right.$$

Exercises 8

1. Let A square metres be the area covered by ripples after t seconds.

$\frac{dA}{dt} = 2\pi t$, so $A = \pi t^2 + c$.

Now when $t = 0$, $A = 0$ and hence $c = 0$.

So the area covered by ripples after t seconds is πt^2 m^2.

After 3 seconds the area covered is 9π m^2.

2. Let N be the size of the population at time t.

Then $\frac{dN}{dt} = 500\pi\cos 2\pi t$.

Hence $N = \int 500\pi \cos 2\pi t dt$

$= 500\pi \cdot \frac{1}{2\pi}\sin 2\pi t + c$

$= 250\sin 2\pi t + c.$

Now when $t = 0$, $N = 3000$, so $3000 = 0 + c$.

Thus $N = 250\sin 2\pi t + 3000$.

Since $\sin 2\pi t$ varies between -1 and 1 and has period 1, the maximum size of the herd is 3250 and the minimum is 2750.

3. Let the height of the plant after t days be h cm.

We are told that $\frac{dh}{dt} = \frac{1}{\sqrt{t+4}}$.

So $h = \displaystyle\int \frac{1}{\sqrt{t+4}}dt = 2\sqrt{t+4} + c.$

When $t = 0$, $h = 20$, so $20 = 2\sqrt{(4)} + c$ which gives $c = 16$.

Thus $h = 2\sqrt{t+4} + 16$.

When $t = 12$, $h = 2\sqrt{16} + 16 = 8 + 16 = 24$,

so after 12 days, the height of the plant is 24 cm.

4. Let C be the concentration of the drug after t hours.

Then $\frac{dC}{dt} = 5e^{-t} - e^{-0.2t}$.

So $C = -5e^{-t} + \frac{1}{0.2}e^{-0.2t} + c$
$\quad = -5e^{-t} + 5e^{-0.2t} + c$.

Now when $t = 0$, $C = 0$, so $0 = -5 + 5 + c$, i.e. $c = 0$.

Hence $C = 5(e^{-0.2t} - e^{-t})$.

To find when the concentration is greatest, we find when $\frac{dC}{dt} = 0$

i.e. $5e^{-t} = e^{-0.2t}$ i.e. $5 = e^{0.8t}$.

Approximate solution to this is $t = 2.01$.

A check shows that $\frac{dC}{dt}$ is positive when $t < 2.01$ and negative when $t > 2.01$.

So the concentration is greatest approximately 2 hours after the drug is administered.

5. Let the distance of the object from the point after t seconds be D metres.

Now speed is rate of change of position, so $\frac{dD}{dt}$ will represent the speed.

However, the object is moving *towards* the fixed point, so D is decreasing and therefore $\frac{dD}{dt}$ is negative.

Hence $\frac{dD}{dt} = -\frac{1}{t+1}$.

So $D = -\displaystyle\int \frac{1}{t+1}dt = -\ln(t+1) + c$.

Now when $t = 0$, $D = 2$, so $2 = -\ln 1 + c$, i.e. $c = 2$.

Thus $D = 2 - \ln(t+1)$.

Now $D = 0$ (i.e. the object is at the fixed point) when $2 = \ln(t+1)$

i.e. $t = e^2 - 1 = 6.4$ (approx.).

So the object will reach the point after approximately 6.4 seconds.